Chaos edges of *z*-logistic maps: Connection between the relaxation and sensitivity entropic indices

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Chaos thresholds of the z-logistic maps $x_{t+1}=1-a|x_t|^z$ ($z>1$; $t=0,1,2,\ldots$) are numerically analyzed at accumulation points of cycles 2, 3, and 5 (three different cycles 5). We verify that the nonextensive q-generalization of a Pesin-like identity is preserved through averaging over the entire phase space. More precisely, we computationally verify $\lim_{t\to\infty}\langle S_{q_{sen}^{av}}\rangle(t)/t=\lim_{t\to\infty}\langle \ln_{q_{sen}^{av}}\xi\rangle(t)/t\equiv\lambda_{q_{sen}^{av}}^{av}$, where the entropy $S_q\equiv(1-\Sigma_i p_i^q)/(q-1)$ ($S_1=-\Sigma_i p_i \ln p_i$), the sensitivity to the initial conditions $\xi\equiv\lim_{\Delta x(0)\to 0}\Delta x(t)/\Delta x(0)$, and $\ln_q x\equiv(x^{1-q}-1)/(1-q)$ ($\ln_1 x=\ln x$). The entropic index $q_{sen}^{av}<1$, and the coefficient $\lambda_{q_{sen}^{av}}^{av}>0$ depend on both z and the cycle. We also study the relaxation that occurs if we start with an ensemble of initial conditions homogeneously occupying the entire phase space. The associated Lebesgue measure asymptotically decreases as $1/t^{1/(q_{rel}-1)}$ ($q_{rel}>1$). These results (i) illustrate the connection (conjectured by one of us) between sensitivity and relaxation entropic indices, namely, $q_{rel}-1\simeq A_n(1-q_{sen}^{av})^{\alpha_n}$, where the positive numbers (A_n,α_n) depend on the cycle; (ii) exhibit an unexpected scaling, namely, $q_{sen}^{av}(\text{cycle } n)=B_n q_{sen}^{av}(\text{cycle } 2)+\epsilon_n$.

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Boltzmann-Gibbs (BG) entropy and the corresponding statistical mechanics generically require strong chaos for their applicability and (well-known) usefulness. This type of requirement was first used in 1872 by Boltzmann himself [1]. Indeed, his "molecular chaos hypothesis" allowed him to arrive at the celebrated distribution of energies at thermal equilibrium now known as the Boltzmann weight. Today, we know that this requirement essentially amounts, for classical nonlinear dynamical systems, to having at least one positive Lyapunov exponent. In the case of many-body Hamiltonian systems, such a condition is satisfied when the interactions are short ranged. Such systems typically exhibit three basic exponential functions [2], namely, (i) the sensitivity to the initial conditions diverges exponentially with time, (ii) physical quantities exponentially relax with time to their value at the stationary state (thermal equilibrium), and (iii) at thermal equilibrium, the probability of a given microstate exponentially decays with the energy of the microstate. These three exponentials of different, though related, nature can be summarized in the following differential equation:

$$dy/dx=a_1 y \quad [y(0)=1], \tag{1}$$

whose solution is $y=e^{a_1 x}$ (the subindex 1 will become transparent soon). Let us make explicit the point. The first physical interpretation concerns the sensitivity to the initial conditions of, say, a one-dimensional case and is defined as

$$\xi(t)\equiv\lim_{\Delta x(0)\to 0}\Delta x(t)/\Delta x(0), \tag{2}$$

where $\Delta x(t)$ is the distance, in phase space, between two copies at time t. If the system has a positive Lyapunov exponent λ_1, then ξ diverges as $\xi=e^{\lambda_1 t}$. In other words, in this case we have $(x,y,a_1)\equiv(t,\xi,\lambda_1)$. The second physical interpretation concerns the relaxation of some (characteristic) physical quantity $O(t)$ to its value $O(\infty)$ at thermal equilibrium. With the definition $\Omega\equiv\frac{O(t)-O(\infty)}{O(0)-O(\infty)}$, we typically have $\Omega=e^{-t/\tau}$, where τ is the relaxation time. In other words, in this case we have $(x,y,a_1)\to(t,\Omega,-1/\tau)$. This relaxation occurs precisely because of the sensitivity to initial conditions, which guarantees strong chaos. Krylov was apparently the first to realize [3], over half a century ago, this deep connection. The third physical interpretation is given by $p_i=e^{-\beta E_i}/Z$ (with $Z\equiv\Sigma_{j=1}^{W}e^{-\beta E_j}$), where E_i is the eigenvalue of the ith quantum state of the Hamiltonian (with its associated boundary conditions) and p_i is the probability of occurrence of the i-th state when the system is in equilibrium with a thermostat whose temperature is $T\equiv 1/k\beta$ (Gibbs' canonical ensemble). In other words, in this case we have $(x,y,a_1)\to(E_i,Zp_i,-\beta)$.

A substantially different situation occurs when the maximal Lyapunov exponent vanishes. In this case the typical differential equation becomes

$$dy/dx=a_q y^q \quad [y(0)=1;\ q\in\mathcal{R}], \tag{3}$$

whose solution is $y=e_q^{a_q x}$, the *q-exponential* function being defined as follows: $e_q^x\equiv[1+(1-q)x]^{1/(1-q)}$ if the quantity between brackets is nonnegative, and zero otherwise ($e_1^x=e^x$). The sensitivity to the initial conditions is given in this case by [4,5] $\xi=e_{q_{sen}}^{\lambda_{q_{sen}}t}$ (*sen* stands for *sensitivity*; this expression stands in fact for the upper bound of ξ). In other words, we have $(x,y,q,a_q)\equiv(t,\xi,q_{sen},\lambda_{q_{sen}})$. The relaxation is typically expected [6] to be characterized by $\Omega=e_{q_{rel}}^{-t/\tau_{q_{rel}}}$ (*rel* stands for *relaxation*). In other words, in this case we have $(x,y,q,a_q)\equiv(t,\Omega,q_{rel},-1/\tau_{q_{rel}})$. For the long-standing metastable states [7] that precede thermal equilibrium for long-range interacting Hamiltonians, it is expected that [8] $p_i=e_{q_{stat}}^{-\beta_{q_{stat}}E_i}/Z_{q_{stat}}$ (with $Z_{q_{stat}}\equiv\Sigma_{j=1}^{W}e_{q_{stat}}^{-\beta_{q_{stat}}E_j}$) (*stat* stands for *stationary*). In other words, in this case we have

$(x,y,q,a_q) \equiv (E_i, Z_{q_{stat}} p_i, q_{stat}, -\beta_{q_{stat}})$. For systems that are at, or close to, the edge of chaos we typically have $q_{sen} \leqslant 1$, $q_{rel} \geqslant 1$, and $q_{stat} \geqslant 1$. For the BG case, where there are one or more positive Lyapunov exponents, we recover the confluence $q_{sen}=q_{rel}=q_{stat}=1$.

The entire q-triplet should be either measurable or calculable for Hamiltonian (or even more complex) systems. And indeed it has recently been measured in the solar wind [9]. However, the generic relation among these three q indices is still unknown. For dissipative systems such as, say, the z-logistic map, no q_{stat} exists. Therefore, the problem reduces to only two q indices, namely, q_{sen} and q_{rel}. Their generic relation also is unknown. In the present paper, we provide numerical evidence of such a connection.

Before entering into the details of the present calculation, let us briefly review the connection with the entropy S_q, the basis of a current generalization of BG statistical mechanics referred to as nonextensive statistical mechanics [10]. This entropy is defined as follows:

$$S_q \equiv \frac{1-\sum_{i=1}^{W} p_i^q}{q-1} = \sum_{i=1}^{W} p_i \ln_q(1/p_i) \tag{4}$$

where the q-*logarithm* function, inverse of the q-exponential, is defined as $\ln_q x \equiv \frac{x^{1-q}-1}{1-q}$ $(\ln_1 x = \ln x)$ and $S_1 = S_{BG} \equiv -\sum_{i=1}^{W} p_i \ln p_i$.

If we partition the phase space of a one-dimensional map (at its edge of chaos) into W small intervals, randomly place N initial conditions into one of those windows, and then run the dynamics for each of those N points, we get, as time t evolves, an occupancy characterized by $\{N_i(t)\}$ $[\sum_{i=1}^{W} N_i(t) = N]$. With $p_i(t) \equiv N_i(t)/N$ we can calculate $S_q(t)$ for any value of q. From this, we can calculate the entropy production per unit time [11], defined as follows:

$$K_q \equiv \lim_{t\to\infty} \lim_{W\to\infty} \lim_{N\to\infty} S_q(t)/t. \tag{5}$$

It has been proved [12] that only $K_{q_{sen}}$ is finite ($K_q=0$ for $q > q_{sen}$ and K_q diverges for $q < q_{sen}$). Furthermore, if we consider the upper bound of $K_{q_{sen}}$ with regard to the choice of the little window within which we put the N initial conditions, we obtain the Pesin-like identity $K_{q_{sen}} = \lambda_{q_{sen}}$. Several aspects of this problem have already been verified for various one-dimensional unimodal maps [13–16], as well as for a two-dimensional conservative map [17]. The influence of averaging was recently studied [18]. It was verified that, while the q-generalized Pesin-like identity is preserved, the value of q_{sen} is changed into q_{sen}^{av} (*av* stands for *average*). The main goal of the present paper is to exhibit that a simple relation exists between q_{sen}^{av} and q_{rel} by making use of the z-logistic map family defined as

$$x_{t+1} = 1 - a|x_t|^z \tag{6}$$

where $(z>1; 0<a\leqslant 2; |x_t|\leqslant 1; t=0,1,2,\ldots)$.

In our simulations, we first check whether the q_{rel} values corresponding to cycles 3 and 5 (denoted 5a, 5b, and 5*c*) are different from those of cycle 2 obtained in [6]. To accomplish this task, we analyze the rate of convergence to the critical attractor when an ensemble of initial conditions is uniformly distributed over the entire phase space (the phase space is partitioned into N_{box} cells of equal size) and we found that, for all cycles that we studied, the volume $W(t)$ occupied by the ensemble exhibits a power-law decay with the same exponent value for given z. As an example, the case of cycle 3 for $z=2$ is given in Fig. 1. The same kind of behavior is obtained for other z values and cycles, which exhibits that q_{rel} does not depend on the cycle (see also Table I). It is worth mentioning at this point that very recent extensive calculations [19] showed that the q_{rel} values should indeed be slightly larger than the values reported in [6]. But, since this is just a systematic shift for all z values, this would have no relevant effect for any scaling function based on q_{rel} values. Then we concentrate on the ensemble averages of the sensitivity function $\xi(t)$ by considering two very close points [throughout this work we take $\Delta x(0)=10^{-12}$] and calculating its value from Eq. (2). This procedure is repeated many times with different values of x randomly chosen in the entire phase space. Finally an average is taken over all $\ln_q \xi(t)$ values. For cycles 3, 5a, 5b, and 5c, we obtain the behavior of $\langle \ln_q \xi \rangle(t)$ as a function of t, for various values of z. We then deduce q_{sen}^{av} by identifying the linear time dependence as illustrated in Fig. 2. We verify that the q_{sen}^{av} and $\lambda_{q_{sen}^{av}}^{av}$ values depend on both z and the cycle, whereas q_{rel} only depends on z. Finally, to investigate the entropy production for the cycles 3, 5a, 5b, and 5c, we use the same procedure as in [15] for cycle 2 of the z-logistic maps. It is numerically verified that, as seen for a typical case in Fig. 3, for each value of z, and for each cycle, the finite entropy production per unit time occurs only for a special value of q which precisely coincides with the one obtained from the sensitivity function. In addition to this, we obtained $K_{q_{sen}^{av}}^{av} = \lambda_{q_{sen}^{av}}^{av}$, which clearly broadens the region of validity of the usual Pesin-like identity [20].

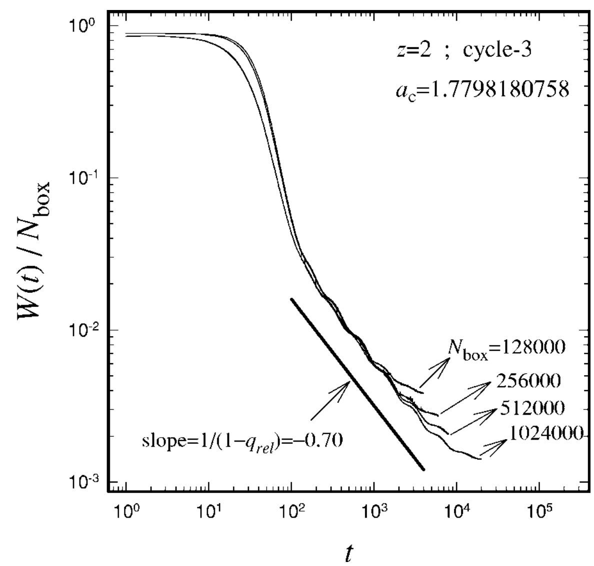

FIG. 1. Occupied volume as a function of discrete time. After a transient period, which is the same for all N_{box} values, the power-law behavior emerges. For each case, the evolution of a set of $10N_{box}$ copies of the system is followed.

TABLE I. z-logistic map family for cycles 2, 3, and 5a.

z	Cycle	a_c	q_{sen}^{av}	q_{rel}	$\lambda_{q_{sen}^{av}}^{av}$	$K_{q_{sen}^{av}}^{av}$
1.75	2	1.355060…	0.37±0.01	2.25±0.02	0.26±0.01	0.26±0.02
1.75	3	1.747303…	0.92±0.01	2.25±0.02	0.48±0.01	0.47±0.02
1.75	5a	1.607497…	0.96±0.01	2.25±0.02	0.42±0.01	0.40±0.02
2	2	1.401155…	0.36±0.01	2.41±0.02	0.27±0.01	0.27±0.02
2	3	1.779818…	0.88±0.01	2.41±0.02	0.49±0.01	0.48±0.02
2	5a	1.631019…	0.93±0.01	2.41±0.02	0.42±0.01	0.40±0.02
2.5	2	1.470550…	0.34±0.01	2.70±0.02	0.28±0.01	0.28±0.02
2.5	3	1.828863…	0.82±0.01	2.70±0.02	0.48±0.01	0.47±0.01
2.5	5a	1.669543…	0.88±0.01	2.70±0.02	0.38±0.01	0.37±0.01
3	2	1.521878…	0.32±0.01	2.94±0.02	0.29±0.02	0.29±0.03
3	3	1.862996…	0.78±0.01	2.94±0.02	0.44±0.01	0.44±0.01
3	5a	1.699440…	0.84±0.01	2.94±0.02	0.34±0.01	0.35±0.01
5	2	1.645533…	0.28±0.01	3.53±0.03	0.30±0.02	0.30±0.03
5	3	1.931072…	0.68±0.01	3.53±0.03	0.36±0.01	0.37±0.01
5	5a	1.773088…	0.73±0.01	3.53±0.03	0.27±0.01	0.25±0.02

Finally, for all cycles that we studied, we numerically verified that a simple scaling relation exists between q_{sen}^{av} and q_{rel} (see Fig. 4), namely,

$$q_{rel}(\text{cycle } n) - 1 \simeq A_n[1 - q_{sen}^{av}(\text{cycle } n)]^{\alpha_n} \quad (7)$$

where $n=2,3,5a,5b,5c$, the values of (A_n, α_n) being given in the caption of Fig. 4; both numbers depend on the cycle. For example, $\alpha_n \simeq 5.1$, $n=2$, and quickly approaches zero when the cycle increases; A_n also decreases when the cycle increases. This kind of relation between these two q indices is seen here in a model system. It is clearly consistent with the confluence expected for BG systems. This is to say, when there is a positive Lyapunov exponent, we obtain $q_{rel}=q_{sen}^{av}=1$.

We also notice (see Fig. 5) an unexpected scaling behavior, namely,

$$q_{sen}^{av}(\text{cycle } n) = B_n q_{sen}^{av}(\text{cycle } 2) + \epsilon_n, \quad (8)$$

with (ϵ_n, B_n) given in the caption of Fig. 5.

Summarizing, we have discussed a paradigmatic family of one-dimensional dissipative maps, and have shown that its (averaged) sensitivity to the initial conditions and its relaxation in phase space follow a simple path. This path is consistent with current nonextensive statistical mechanical concepts, and considerably extends the validity of Pesin-like identities. The sensitivity to the initial conditions is characterized by $q_{sen}^{av}<1$, which monotonically approaches unity with increasing cycle size (at least for the specific cycles that we have studied here), and decreases with z. It is further

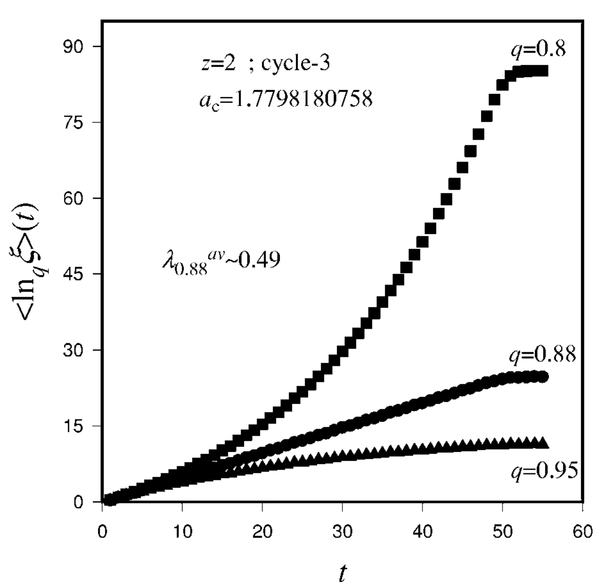

FIG. 2. Time dependence of $\langle \ln_q \xi \rangle$.

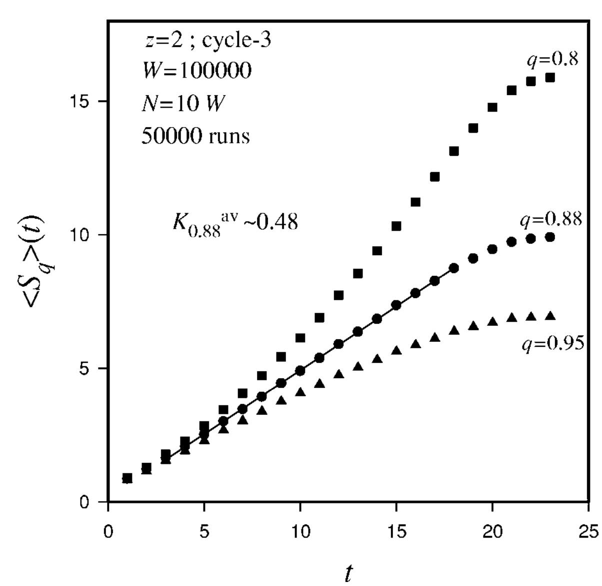

FIG. 3. Time dependence of $\langle S_q \rangle$.

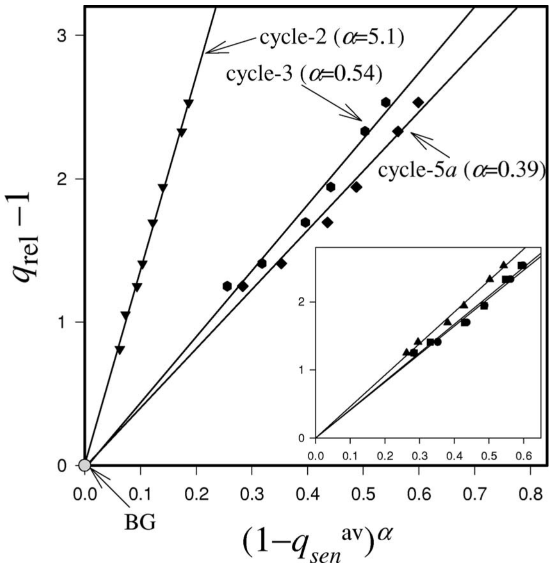

FIG. 4. Straight lines: q_{rel}(cycle 2)$-1=13.5[1-q_{sen}^{av}$ (*cycle* 2)$]^{5.1}$, q_{rel}(cycle 3)$-1=4.6[1-q_{sen}^{av}$(*cycle* 3)$]^{0.54}$, and q_{rel}(cycle 5a)$-1=4.1[1-q_{sen}^{av}$(cycle 5a)$]^{0.39}$. Inset: All three cycles 5 are shown together: [q_{rel}(cycle 5b)$-1=4.2[1-q_{sen}^{av}$ (cycle 5b)$]^{0.30}$, and q_{rel}(cycle 5c)$-1=4.7[1-q_{sen}^{av}$(cycle 5c)$]^{0.23}$]. Notice that the Boltzmann-Gibbs limiting case appears as a special point attained for all cycles studied here.

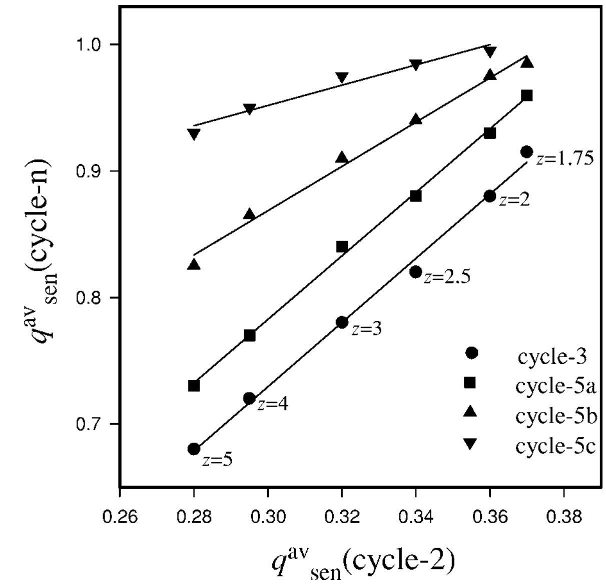

FIG. 5. Straight lines: q_{sen}^{av}(cycle 3)$=2.5q_{sen}^{av}$(cycle 2)-0.03, q_{sen}^{av}(cycle 5a)$=2.5q_{sen}^{av}$(cycle 2)$+0.03$, q_{sen}^{av}(cycle 5b)$=1.75q_{sen}^{av}$ (cycle 2)$+0.34$, and q_{sen}^{av}(cycle 5c)$=0.8q_{sen}^{av}$(cycle 2)$+0.71$.

characterized by $\lambda_{q_{sen}^{av}}^{av}$, which exhibits a maximum as a function both of the cycle size and of z. The relaxation is characterized by $q_{rel}>1$, which monotonically increases with z, and does not depend on the cycle. This numerical study has enabled us to exhibit two interesting relations, namely, Eqs. (7) and (8). These results are expected to illuminate, among other things, the case of long-range-interacting Hamiltonian systems, the situation of which is even more complex since a third entropic index q_{stat} is expected (which would characterize the energy distribution at metastable states). Analytic approaches to the present scalings are certainly most welcome.

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